Maximal Regular Extension of Convex Risk Measure

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A General Quastion

Convex risk measures

- well-studied on L^∞ in various aspects:
  Delbaen, Föllmer, Schied, Kusuoka, Jouini, Schachermayer, Touzi, Kupper,...
- A direction: From bounded to unbounded, e.g.,
  - L^p: Kaina/Rüschendorf, Acciaio; Orlicz (type) spaces: Cheridito/Li, Arai
  - Abstract Fréchet lattices: Biagini/Frittelli ...

A "General" Question

How far can we go beyond L^∞?

- usual flow: given a space $\mathscr{X}$, what ρ's are possible?
- ours: given a ρ on L^∞, what spaces are possible?
  - Starting from a "regular" risk measure ρ^0 on L^∞,
  - extend ρ^0 to as big space $\mathscr{X}$ as possible preserving the regularity.

What is the maximal such $\mathscr{X}$? exists?

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Some Definitions

Convex Risk Measure

$\mathscr{X} \subset L^0$: vector space.

- $\rho : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ is a convex risk measure if
  (R1) convex: $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$ if $\alpha \in (0,1)$;
  (R2) monotone: $X \leq Y$ a.s. $\Rightarrow \rho(X) \geq \rho(Y)$;
  (R3) cash-invariant: $\rho(X + \alpha) = \rho(X) - \alpha$ if $\alpha \in \mathbb{R}$.
  Just a matter of taste! But we prefer convex & increasing functions.
- A couple of normalization:
  (R4) $\rho(0) = 0$;
  (R5) $\mathbb{P}(A) > 0 \Rightarrow \rho(\alpha 1_A) > 0$ for all $\alpha > 0$.
- If $\mathscr{X} = L^\infty$, (R2,3) show $-\|X\|_\infty \leq \rho(X) \leq \|X\|_\infty$, hence finite-valued.
- A vector space $\mathscr{X} \subset L^0$ is solid $\overset{\text{def}}{\Leftrightarrow}$ $|X| \leq |Y|$ & $Y \in \mathscr{X} \Rightarrow X \in \mathscr{X}$.
  - L^0 is a Riesz space (vector lattice) with the order "$\leq$, a.s."
  - If $\mathscr{X}$ is solid, then $|X| \in \mathscr{X}$, hence a lattice on its own.

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Regularity Properties: Lebesgue vs Fatou

$$\text{Fatou} : X_n \to X \text{ a.s.}, |X_n| \le |Y|, \exists Y \in \mathscr{X} \Rightarrow \rho(X) \le \liminf_n \rho(X_n).$$
$$\text{Lebesgue} : X_n \to X \text{ a.s.}, |X_n| \le |Y|, \exists Y \in \mathscr{X} \Rightarrow \rho(X) = \lim_n \rho(X_n).$$

- If $\mathscr{X} = L^\infty$, "$|X_n| \le |Y|$ & $Y \in L^\infty$" $\Leftrightarrow \sup_n \|X_n\|_\infty < \infty$, thus
  - Fatou (resp. Lebesgue) $\Leftrightarrow$ weak*-lsc (resp. continuity).
  - Lebesgue on $L^\infty \Rightarrow$ Fatou on $L^\infty \Rightarrow$ "good" dual representation:

$$\rho(X) = \sup_{Q \ll \mathbb{P}} (E_Q[X] - \gamma(Q)), \quad \gamma(Q) := \sup_{X \in L^\infty} (E_Q[X] - \rho(X))$$

  - Lebesgue seems strong, but at least on L^∞,

(Special Case of) Delbaen (2009, Math. Finance)

If $\rho : L^\infty \to \mathbb{R}$ has a finite-valued extension to L^p, $\exists p < \infty$ (and $(\Omega, \mathcal{F}, \mathbb{P})$ atomless), ρ has the Lebesgue property on L^∞.

- "I" don't know any concrete example that violate this!!

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Extension of Convex Risk Function

Definition (Regular Extension)

$\rho : \mathscr{Y} \to \mathbb{R} \cup \{+\infty\}$ is a regular extension of $\rho^0 : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ iff

- $\mathscr{X} \subset \mathscr{Y}$ and $\rho|_{\mathscr{X}} = \rho^0$;
- ρ is Lebesgue on $\mathscr{Y}$.

- Compatibility: $\mathscr{X} \subset \mathscr{Y}$ & Lebesgue on $\mathscr{Y} \Rightarrow$ on $\mathscr{X}$.

Example: $\rho(X) = \log E[\exp(X)]$ (entropic risk measure).

- Obviously, this is well-defined with Fatou on L^0.
- ρ is Lebesgue on $M^{\exp} := \{X : E[\exp(\alpha|X|)] < \infty, \forall \alpha > 0\}$ (Orlicz heart), but
- NOT on $L^{\exp} := \{X : E[\exp(\alpha|X|)] < \infty, \exists \alpha > 0\}$ (Orlicz space).

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Goal of This Talk

Question

Given a convex risk function $\rho^0 : L^\infty \to \mathbb{R}$ with the Lebesgue prop. on L^∞,

What is the maximal solid $\mathscr{X} \subset L^0$ to which ρ^0 has a regular extension?

- Exists? Unique?? What kind of the space???

Our "philosophy" on the study of risk measures on $\mathscr{X} \supsetneq L^\infty$

- Usual way: reconstruct a whole theory exactly as did in L^∞:
  - characerizing the "order-continuous dual" $\mathscr{X}_n^\sim$,
  - as well as the topology $\sigma(\mathscr{X}, \mathscr{X}_n^\sim)$ ← difficult
  - checking $\sigma(\mathscr{X}, \mathscr{X}^\sim)$-lsc, obtaining dual representation......
- But risk measure is "something like a measure",
  - its structure "should be" determined on L^∞,
  - we want to obtain "maximal" results with "minimal" effort.

Why Lebesgue? Why not Fatou?

Lebesgue property

- justifies "approximation by bounded r.v.'s"

$$\rho(X) \stackrel{?}{=} \lim_N \rho(X1_{\{|X| \leq N\}}) \stackrel{??}{=} \lim_n \lim_m \rho(((-n) \vee X) \wedge m)$$

  - Mathematically, this guarantees uniqueness of extension.
  - Practically, provide a big toolbox available for computation.
- is stable under inf-convolution:

$$\rho \square f(X) = \inf_Y \{\rho(X - Y) + f(Y)\}$$

  - Risk indifference pricing, asset alocation.
- provide a "good" duality for robust utility maximzation/indifference price:

$$\sup_\theta \inf_{P \ll \mathbb{P}} (E_P[U(\theta \cdot S_T + B)] + \gamma(P)) \stackrel{??}{=} \inf_Q \text{"certain functional of } Q\text{"}$$

  - Relatively easy if Q is allowed to be non-σ additive.
  - Need a kind of compactness ($\Leftarrow$ Lebesgue enough, Fatou is not enough)

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A Simple Observation

Very simple but crucial observation

Let $\mathscr{X} \subset L^0$ be a solid subspace, and $\rho : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ given

- If ρ is Lebesgue on $\mathscr{X}$, it must satisfy

$$\lim_{N\to\infty} \rho\left(\alpha|X|1_{\{|X|>N\}}\right) = 0, \quad \forall\alpha > 0.$$

  - $|\alpha X 1_{\{|X|>N\}}| \leq \alpha|X| \in \mathscr{X}$ & $X \in \mathscr{X} \Rightarrow \alpha X 1_{\{|X|>N\}} \in \mathscr{X}$ (since solid)
  - and obviously $\alpha X 1_{\{|X|>N\}} \to 0$ a.s.
  - In particular, ρ must be finite valued on $\mathscr{X}$: $-\rho(-|X|) \leq \rho(|X|) \leq \frac{1}{2}(\rho(1|X|1_{\{|X|>N\}}) + N)$ for some large N.

- Thus ρ^0 can not be regularly extended beyond

$$“\left\{X \in L^0 : \lim_{N\to\infty} “\rho^0”(\alpha|X|1_{\{|X|>N\}}) = 0,\ \forall\alpha > 0\right\}” \quad \leftarrow \text{only formal at now}$$

  - If “$\rho^0 : L^0_+ \to \mathbb{R}$” is well-defined, this space is solid (clear) & linear since

$$|X+Y|1_{\{|X+Y|>N\}} \leq (|X|+|Y|)\left(1_{\{|X|+|Y|>N\}\cap\{|X|>|Y|\}} + 1_{\{|X|+|Y|>N\}\cap\{|X|\leq|Y|\}}\right)$$
$$\leq 2|X|1_{\{|X|>N/2\}} + 2|Y|1_{\{|Y|>N/2\}}$$

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Suppose $\rho^0 : L^\infty \to \mathbb{R}$ has Lebesuge (hence Fatou) then,

$$\rho^0(X) = \sup_{Q \in \mathcal{Q}_\gamma} (E_Q[X] - \gamma(Q)), \quad X \in L^\infty$$

$$\gamma(Q) := \sup_{X \in L^\infty} (E_Q[X] - \rho^0(X)), \; \mathcal{Q}_\gamma := \{Q \ll \mathbb{P} : \gamma(Q) < \infty\}$$

- This expression makes sense far beyond L^∞. Define

$$\hat{\rho}(X) := \sup_{Q \in \mathcal{Q}_\gamma} (E_Q[X] - \gamma(Q)), \; X \in \mathcal{D}_\gamma := \{X \in L^0 : \; X^- \in \bigcap_{Q \in \mathcal{Q}_\gamma} L^1(Q)\}$$

  - $\mathcal{D}_\gamma$ is not linear, but a convex cone with $L^0_+ \subset \mathcal{D}_\gamma$.
  - $\hat{\rho}$ is convex, monotone, cash-invariant (i.e., a "convex risk function" on $\mathcal{D}_\gamma$).
  - $\hat{\rho}|_{L^\infty} = \rho^0$ ($\Rightarrow \hat{\rho}(0) = 0$ and sensitive).

Now the following makes sense

$$M_u^{\hat{\rho}} := \{X \in L^0 : \; \lim_{N \to \infty} \hat{\rho}\left(\alpha |X| 1_{\{|X| > N\}}\right) = 0, \; \forall \alpha > 0\}$$

- From prev. arguments, $L^\infty \subset M_u^{\hat{\rho}}$ and $M_u^{\hat{\rho}}$ is a solid subspace

Suppose $\rho^0 : L^\infty \to \mathbb{R}$ has Lebesuge (hence Fatou) then,

$$\rho^0(X) = \sup_{Q \in \mathcal{Q}_\gamma} (E_Q[X] - \gamma(Q)), \quad X \in L^\infty$$

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- This expression makes sense far beyond L^∞. Define

$$\hat{\rho}(X) := \sup_{Q \in \mathcal{Q}_\gamma} (E_Q[X] - \gamma(Q)), \ X \in \mathcal{D}_\gamma := \{X \in L^0 : \ X^- \in \bigcap_{Q \in \mathcal{Q}_\gamma} L^1(Q)\}$$

  - $\mathcal{D}_\gamma$ is not linear, but a convex cone with $L^0_+ \subset \mathcal{D}_\gamma$.
  - $\hat{\rho}$ is convex, monotone, cash-invariant (i.e., a "convex risk function" on $\mathcal{D}_\gamma$).
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A Norm

$$\|X\| := \inf\{\lambda > 0 : \hat{\rho}(|X|/\lambda) \leq 1\}$$ ← gauge, Minkowski func.

Easy to observe:

- $\|\cdot\|$ is a seminorm ← standard in Orlicz space theory
- $|X| \leq |Y| \Rightarrow \|X\| \leq \|Y\|$, i.e. $\|\cdot\|$ is a lattice seminorm
- $\|X\| < \infty \Leftrightarrow \hat{\rho}(\alpha|X|) < \infty$, $\exists\alpha > 0 \Leftarrow$ "$\cdots$" $\forall\alpha > 0 \Leftarrow X \in M_u^{\hat{\rho}}$.
    - "$\Rightarrow$" clear. "$\Leftarrow$": $\hat{\rho}(\varepsilon\alpha|X|) = \hat{\rho}(\varepsilon\alpha|X| + (1-\varepsilon)0) \leq \varepsilon\hat{\rho}(\alpha|X|) \downarrow 0$.

An Elementary Inequality

$\|X\|_{L^1(Q)} \leq (1 + \gamma(Q))\|X\|$, $\forall Q \ll \mathbb{P}$.

- $\|X\| = 0 \Rightarrow \|X\|_{L^1(Q_0)} = 0 \Leftrightarrow X = 0$ a.s. → $\|\cdot\|$ is a norm
    - (R5) (sensitivity) $\Leftrightarrow \exists Q_0 \sim \mathbb{P}$, $\gamma(Q_0) < \infty$.
- $M_u^{\hat{\rho}} \subset \bigcap_{Q \in \mathcal{Q}_\gamma} L^1(Q) = \mathcal{D}_\gamma \cap (-\mathcal{D}_\gamma) \Rightarrow \hat{\rho}$ is well-defined on $M_u^{\hat{\rho}}$
- Also, $-\infty < \hat{\rho}(X) \leq \hat{\rho}(|X|) < \infty$, hence $\hat{\rho}$ is finite valued on $M_u^{\hat{\rho}}$.

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Main Result

$(\hat\rho, M_u^{\hat\rho})$ is the maximal regular extension

$$M_u^{\hat\rho} = \{X \in L^0 : \lim_N \hat\rho\left(\alpha|X|1_{\{|X|>N\}}\right) = 0,\ \forall\alpha > 0\},\ \|X\| = \inf\{\lambda > 0 : \hat\rho(|X|/\lambda) \le 1\}$$

Theorem 1 ($M_u^{\hat\rho}$ *is the maximum solid space admitting a regular extension*)

1. $(M_u^{\hat\rho}, \|\cdot\|)$ is an order-continuous Banach lattice, i.e.,
   - $(M_u^{\hat\rho}, \|\cdot\|)$ is a Banach space and $\|\cdot\|$ satisfies $|X| \le |Y| \Rightarrow \|X\| \le \|Y\|$,
   - $X \mapsto \|X\|$ has Lebesgue (order-continuous):

$$X_n \to 0 \text{ a.s. } \&\ \exists Y \in M_u^{\hat\rho},\ |X_n| \le |Y| \ \Rightarrow\ \lim_n \|X_n\| = 0. \qquad (*)$$

2. $(\hat\rho, M_u^{\hat\rho})$ is a regular extension of ρ^0, and if $(\rho, \mathscr{X})$ is such ($\mathscr{X}$ solid),

$$\mathscr{X} \subset M_u^{\hat\rho} \text{ and } \rho = \hat\rho \text{ on } \mathscr{X}$$

- Given 1, Lebesgue prop. of $\hat\rho$ follows from an extended Namioka-Klee: $\forall$ *monotone convex function f on a Banach lattice is norm conti. on* $\operatorname{int}\operatorname{dom}(f)$.
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$(\hat\rho, M_u^{\hat\rho})$ is the maximal regular extension

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Some Keys

$\hat{\rho}(\alpha|X|1_{\{|X|>N\}}) \downarrow 0,\ \forall\alpha > 0 \Leftrightarrow \|X1_{\{|X|>N\}}\| \downarrow 0 \Leftrightarrow \|X1_{A_n}\| \downarrow 0\ (\mathbb{P}(A_n) \downarrow 0)$.

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Why subscript "u"?

Analogy to Orlicz-Type Spaces

- $M^{\hat\rho} := \{X \in L^0 : \rho(\alpha|X|) < \infty, \forall \alpha > 0\}$ ← "Orlicz heart".
- $M_u^{\hat\rho} \subset M^{\hat\rho} \subset L^{\hat\rho} := \{X \in L^0 : \rho(\alpha|X|) < \infty, \exists \alpha > 0\}$ ← "Orlicz space"
- Example (entropic case) : $\rho(X) = \log E[\exp(X)]$. In this case

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- A Question: always $M_u^{\hat\rho} \overset{??}{=} M^{\hat\rho}$? NO!!!

$\Omega = \mathbb{N}$, $P_1(\{1\}) = 1$, $P_n(\{1\}) = 1 - \frac{1}{n}$, $P_n(\{n\}) = \frac{1}{n}$, $\mathcal{P} := \overline{\mathrm{conv}}(P_n : n \in \mathbb{N})$

$$\rho(X) = \sup_{P \in \mathcal{P}} E_P[X] = \sup_n E_{P_n}[X] \quad (\text{if } X \geq 0) \quad \leftarrow \text{coherent}$$

$\leftarrow \gamma(Q) = 0$ if $Q \in \mathcal{P}$, $= +\infty$ if not; $\mathcal{P}$ compact ⇔ Lebesgue.

- Take $X(n) = n$. Then
  - $E_{P_n}[X] = 2 - 1/n \Rightarrow \sup_n E_{P_n}[X] = 2 < \infty$ hence $X \in M^{\hat\rho}$.
  - BUT $E_{P_n}[X 1_{\{X>N\}}] = 1_{\{n>N\}} \Rightarrow \sup_n E_{P_n}[X 1_{\{X>N\}}] \equiv 1 \Rightarrow X \notin M_u^{\hat\rho}$

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Theorem 2 ($M^{\hat\rho}_u$ = "$M^{\hat\rho}$ + Uniform Integrability")

For $X \in M^{\hat\rho}$, TFAE:

1. $X \in M^{\hat\rho}_u$;
2. $\{X dQ/d\mathbb{P} : \gamma(Q) \leq \beta\}$ is UI, $\forall \beta > 0$. ← thus "u"
3. $\forall Y \in L^\infty$, $\sup_{Q \in \mathcal{Q}_\gamma}(E_Q[YX] - \gamma(Q))$ is attained.

- 1 ⇒ 2 ⇒ 3: easy; 2 ⇒ 1: a "clever" use of a minimax.
- 3 ⇒ 2: a perturbed James's theorem:

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E: Banach, $f : E \to \mathbb{R} \cup \{+\infty\}$ coercive: $\lim_{\|x\| \to \infty} f(x)/\|x\| = +\infty$. Then

$$\sup_{x \in E}(\langle x, l\rangle - f(x)) \text{ is attained } \forall l \in E^* \Rightarrow \{x : f(x) \leq \alpha\} \text{ is weakly rel. compact}$$

- cf James: weakly closed & bounded $C \subset E$ (Banach) is weakly compact iff

$$\sup_{x \in C}\langle x, l\rangle = \sup_{x \in E}(\langle x, l\rangle - \delta_C(x)) \text{ is attained } \forall l \in E^* \quad (C = \{x : \delta_C(x) \leq 1\})$$

An Application

“Lebesgue=Max=Compact” Theorem

Application: a convenient extension of JST theorem

- Jouini-Schachermayer-Touzi, Delbaen: For $\rho : L^\infty \to \mathbb{R}$ with Fatou, AE:
  1. ρ has the Lebesgue property;
  2. $\{Q : \gamma(Q) \leq \alpha\}$ is weakly compact in L^1;
  3. $\sup_Q(E_Q[X] - \gamma(Q))$ is attained.
- Orihuela-Ruiz Galán12: Same but with $\langle L^\Psi, M^{\Psi^*}\rangle$ instead of $\langle L^\infty, L^1\rangle$.

Corollary: Analogue of JST for solid $\mathscr{X}$ (Dom = UI)

$\rho : \mathscr{X} \to \mathbb{R}$, $\mathscr{X}$ solid, $\rho|_{L^\infty}$ has Fatou. Then AE

1. ρ has Lebesgue on $\mathscr{X}$;
2. $\{XdQ/d\mathbb{P} : \gamma_\infty(Q) \leq \alpha\}$ is UI, $\forall X \in \mathscr{X}$; $\leftarrow \gamma_\infty(Q) = \sup_{X \in L^\infty}(E_Q[X] - \rho(X))$
3. $\sup_{\gamma_\infty(Q)<\infty}(E_Q[X] - \gamma_\infty(Q))$ is attained $\forall X \in \mathscr{X}$.

- All Orlicz spaces & hearts, L^p with $p \in [0, \infty]$ are solid.
- No (weak & weak*) topology of $\mathscr{X}$ involved!! $\leftarrow$ very very hard (for me)!!
- Don't need to understand what's "$\sigma(\mathscr{X}, \mathscr{X}^*)$-compact"!!

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$\rho : \mathscr{X} \to \mathbb{R}$, $\mathscr{X}$ solid, $\rho|_{L^\infty}$ has Fatou. Then AE

1. ρ has Lebesgue on $\mathscr{X}$;
2. $\{XdQ/d\mathbb{P} : \gamma_\infty(Q) \leq \alpha\}$ is UI, $\forall X \in \mathscr{X}$; $\leftarrow \gamma_\infty(Q) = \sup_{X \in L^\infty}(E_Q[X] - \rho(X))$
3. $\sup_{\gamma_\infty(Q)<\infty}(E_Q[X] - \gamma_\infty(Q))$ is attained $\forall X \in \mathscr{X}$.

- All Orlicz spaces & hearts, L^p with $p \in [0, \infty]$ are solid.
- No (weak & weak*) topology of $\mathscr{X}$ involved!! $\leftarrow$ very very hard (for me)!!
- Don't need to understand what's "$\sigma(\mathscr{X}, \mathscr{X}^*)$-compact"!!

Application: a convenient extension of JST theorem

- Jouini-Schachermayer-Touzi, Delbaen: For $\rho : L^\infty \to \mathbb{R}$ with Fatou, AE:
  1. ρ has the Lebesgue property;
  2. $\{Q : \gamma(Q) \le \alpha\}$ is weakly compact in L^1;
  3. $\sup_Q(E_Q[X] - \gamma(Q))$ is attained.
- Orihuela-Ruiz Galán12: Same but with $\langle L^\Psi, M^{\Psi^*}\rangle$ instead of $\langle L^\infty, L^1\rangle$.

Corollary: Analogue of JST for solid $\mathscr{X}$ (Dom $=$ UI)

$\rho : \mathscr{X} \to \mathbb{R}$, $\mathscr{X}$ solid, $\rho|_{L^\infty}$ has Fatou. Then AE

1. ρ has Lebesgue on $\mathscr{X}$;
2. $\{XdQ/d\mathbb{P} : \gamma_\infty(Q) \le \alpha\}$ is UI, $\forall X \in \mathscr{X}$; $\leftarrow \gamma_\infty(Q) = \sup_{X \in L^\infty}(E_Q[X] - \rho(X))$
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- All Orlicz spaces & hearts, L^p with $p \in [0, \infty]$ are solid.
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Final Comments

- Robust representation of ρ on (Fréchet lattice) $\mathscr{X}$: if Fatou,

$$\rho(X) = \sup_{Q \in (\mathscr{X}_n^\sim)_{+,1}} (E_Q[X] - \rho^*(Q)), \quad \rho^*(Q) := \sup_{X \in \mathscr{X}} (E_Q[X] - \rho(X))$$

  - $\mathscr{X}_n^\sim$: order continuous dual of $\mathscr{X}$ $\leftarrow$ not easy!!
    Note: $\mathscr{X} \hookleftarrow L^\infty \Rightarrow \mathscr{X}^* \hookrightarrow ba$: finitely additive signed measures.

- Our regular extension $(\hat{\rho}, M_u^{\hat{\rho}})$ is unique & maximal, in particular,

  If $\rho : \mathscr{X} \to \mathbb{R}$ has Lebesgue (and $\mathscr{X}$ solid),

$$\hat{\rho}(X) = \sup_{Q \in \mathcal{Q}_\gamma} (E_Q[X] - \gamma(Q)) \text{ with } \gamma(Q) = \sup_{X \in L^\infty} (E_Q[X] - \rho(X))$$

  - Don't need to know what's $\mathscr{X}_n^\sim$.

Thank You for Your Attention !!

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